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Distinguishing fractional and white noise in one and two dimensions

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We discuss the link between uncorrelated noise and the Hurst exponent for one- and two-dimensional interfaces. We show that long range correlations cannot be observed using one-dimensional cuts through two-dimensional self-affine surfaces whose height distributions are characterized by a Hurst exponent H lower than $-1/2$. In this domain, fractional and white noise are not distinguishable. A method analyzing the correlations in two dimensions is necessary. For $H>-1/2$, a crossover regime leads to an systematic overestimate of the Hurst exponent.

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Self-affine surfaces are abundant in Nature. They are the bread and butter of quantitative characterization of growth phenomena such as fracture surfaces [1], interface growth, and roughening phenomena [2].

A self-affine surface $h(x,y)$ is defined by its behavior under the scale transformation [3]

$$x \rightarrow \lambda x,$$
$$y \rightarrow \lambda y, \tag{1}$$
$$h \rightarrow \lambda^H h,$$

where H is the Hurst exponent.

Most commonly, the Hurst exponent is in the interval $0 \leq H \leq 1$. For instance, fracture surfaces exhibit a Hurst exponent close to 0.8 [1]. Sea floor topography is self-affine, with a Hurst exponent close to 0.5 [4]. When $H>1$, the surface is no longer asymptotically flat. When $H<0$, the roughness distribution of the surface is referred to as *fractional noise.* Fractional noise is typically encountered in Nature in quantities that depend on the local slope of the topography: mechanical stresses, light scattering, and fluid flow [5,6]. For instance, the stress field on the interface between two rough elastic blocks forced into complete contact is a fractional noise with a Hurst exponent H_σ, related to the Hurst exponent of the rough surface H as $H_\sigma = H-1$ [7].

In this Brief Report we show that, for values of H in the range $[-1,-1/2]$, self-affinity takes on very different character in one and two dimensions. If this difference is ignored, one may obtain incorrect results when analyzing experimental data, no matter what method one uses for estimating H. Numerous tools exist for measuring Hurst exponents in the range $0<H<1$. Few of these methods have been tested systematically in the range $H<0$ [8].

The power spectrum of a self-affine trace $h(x)$, characterized by a Hurst exponent H, is given in one dimension by

$$P(k) \sim \frac{1}{k^{1+2H}} \quad \text{(in one dimension)}, \tag{2}$$

while the power spectrum of a two-dimensional self-affine surface $h(x,y)$, characterized by the same Hurst exponent, is

$$P(k) \sim \frac{1}{k^{2+2H}} \quad \text{(in two dimensions)}. \tag{3}$$

White, i.e., uncorrelated noise, has a constant power spectrum both in one and two dimensions. Consequently, the value of the Hurst exponent H which describes white noise in one dimension is obtained from Eq. (2),

$$H_{wn} = -\frac{1}{2}, \tag{4}$$

while from Eq. (3), for the two-dimensional case we find

$$H_{wn} = -1. \tag{5}$$

This result is unexpected. One would have expected the value of the Hurst exponent corresponding to white noise to be independent of dimension.

This result is even more paradoxical when we analyze *cuts* through a two-dimensional self-affine surface. Suppose one is given a two-dimensional surface with a Hurst exponent $H=-1/2$, and is asked to determine H. Analyzing the *two-dimensional* power spectrum of this surface will lead to $P(k) \sim 1/k$ — a $1/f$ spectrum, while analyzing the power

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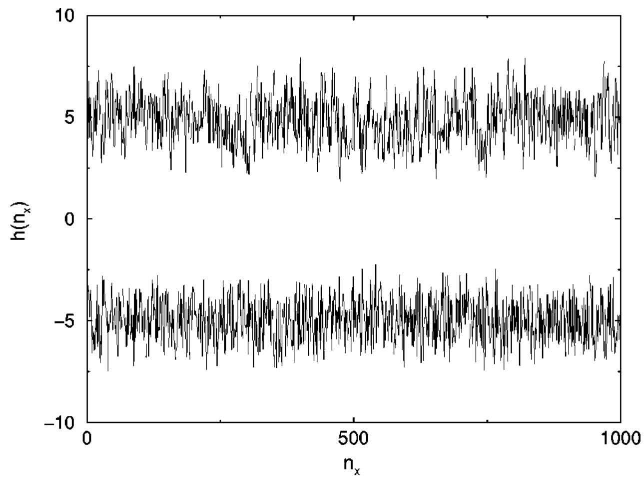

FIG. 1. One-dimensional cut through a two-dimensional self-affine surface with $H=-1/2$ (upper curve) and $H=-1.0$ (lower curve).

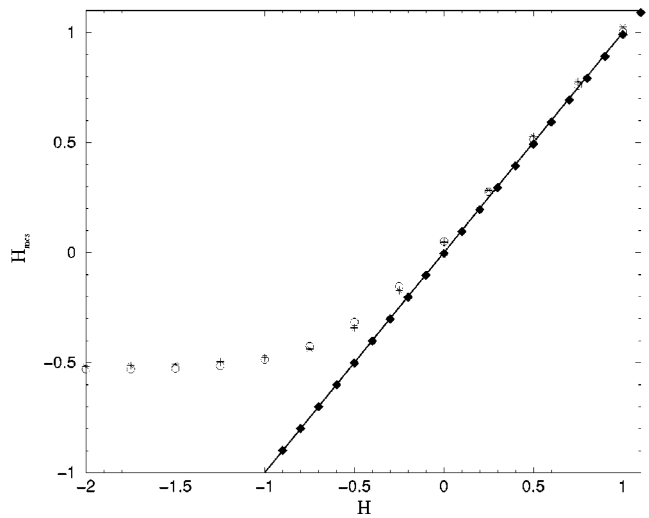

FIG. 3. Measured Hurst exponent H_{mes} vs Hurst exponent H for two-dimensional surfaces. Circles are based on power spectra measurements along one-dimensional cuts, stars are based on AWC analysis along one-dimensional cuts, and filled lozenges are based on two-dimensional power spectra measurements.

spectrum of *one-dimensional cuts* through the surface yields white noise. We illustrate this point in Fig. 1 where we show one-dimensional cuts through two-dimensional surfaces with $H=-1/2$ and -1, respectively. The synthetic surfaces were generated using a Fourier technique [4,9]. In Fig. 2 we show their one-dimensional power spectra.

Yet a third problem is seen when analyzing a two-dimensional self-affine surface with a Hurst exponent in the range $-1\leqslant H\leqslant -1/2$. Analyzing the correlations in the surface using the two-dimensional power spectrum yields the correct value $-1\leqslant H\leqslant -1/2$. However, analyzing one-dimensional cuts through the two-dimensional surface using the one-dimensional power spectrum method or the average wavelet coefficient (AWC) method [10,11] yields the *constant* value $H=-1/2$. This is illustrated in Fig. 3. On the other hand, analyzing one-dimensional traces generated with the Hurst exponent in the range $-1\leqslant H\leqslant -1/2$ yields the input value of H. This is illustrated in Fig. 4. When comparing Fig. 5, showing a one-dimensional self-affine trace with $H=-1$ with a one-dimensional cut through a two-dimensional self-affine surface with the same Hurst exponent, we see a clear difference between the two traces. It is this difference that the different measuring methods pick up.

This unexpected situation was recently encountered in the analysis of the stress field of elastic self-affine surfaces in full contact [7]. As mentioned above, if the elastic surfaces are characterized by a Hurst exponent H, the corresponding stress field has a Hurst exponent $H_\sigma = H-1$. However, when

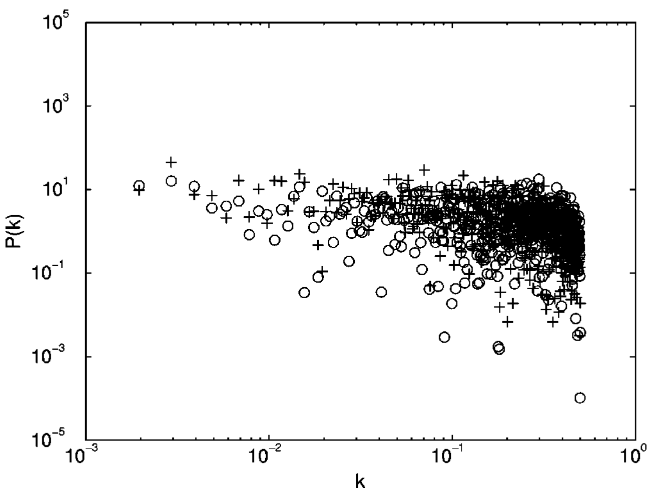

FIG. 2. One-dimensional power spectra of the two self-affine curves of Fig. 1. Circles refer to $H=-1.0$, and plus signs to $H=-0.5$.

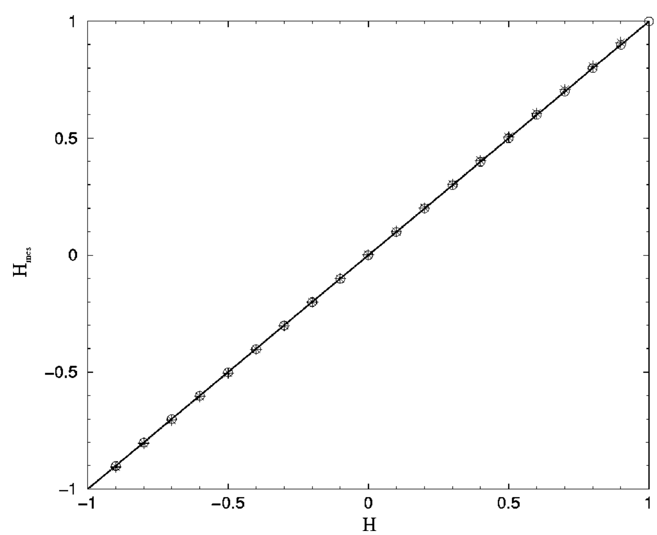

FIG. 4. Measured Hurst exponent H_{mes} vs Hurst exponent H for one-dimensional traces. Circles are based on power spectra measurements in one dimension, and stars are based on AWC analysis in one dimension.

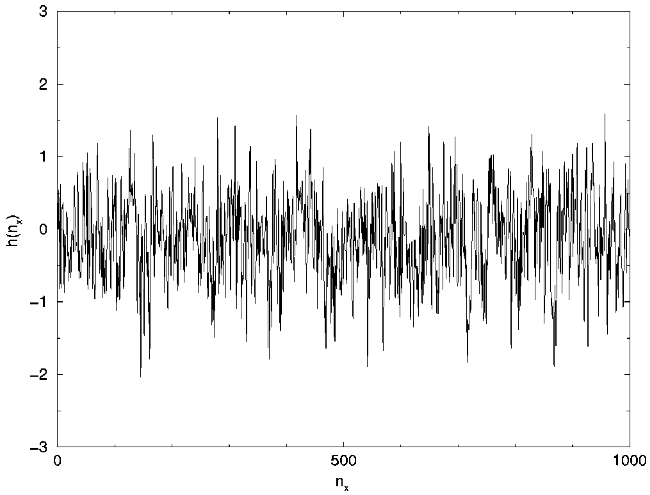

FIG. 5. A one-dimensional self-affine trace with $H=-1$.

analyzing the stress using one-dimensional cuts, H_σ always saturated at the value $-1/2$ as H was lowered to values below 1/2.

In order to understand what lies behind this unexpected behavior, we need a model self-affine surface that is accessible to analytical calculations. The model we choose is based on the Fourier method to generate self-affine surfaces.

We discretize the surface, assuming it to be $h(n_x,n_y)$, where $0\leqslant n_x\leqslant N-1$ and $0\leqslant n_y\leqslant N-1$ are the positions of the nodes on a two-dimensional square lattice. The surface may be represented in Fourier space as

$$h(n_x,n_y)=\sum_{k_x=0}^{N-1}\sum_{k_y=0}^{N-1} e^{(2\pi i/N)[k_x n_x+k_y n_y]}\frac{\eta(k_x,k_y)}{(k_x^2+k_y^2)^{(H+1)/2}}, \tag{6}$$

where $\eta(n_x,n_y)$ is a white (Gaussian) noise defined by a zero mean and a second moment satisfying

$$\langle \eta(k_x,k_y)\,\eta(k_x',k_y')\rangle = 2D\,\delta_{k_x,k_x'}\delta_{k_y,k_y'}. \tag{7}$$

We see immediately from Eq. (6) that for $H=-1$, $h(n_x,n_y)$ is white noise, as we are then Fourier transforming the white noise $\eta(k_x,k_y)$ directly.

A one-dimensional self-affine trace, on the other hand, may be written

$$h(n_x)=\sum_{k_x=0}^{N-1} e^{(2\pi i/N)k_x n_x}\frac{\eta(k_x)}{k_x^{H+1}}, \tag{8}$$

where $\eta(k_x)$ is again white noise.

In order to study a one-dimensional cut through the two-dimensional surface $h(n_x,n_y)$, we place the cut along the x axis and Fourier transform $h(n_x,n_y)$ in the x direction only. This gives us

$$\tilde{h}(k_x,n_y)=\sum_{k_y=0}^{N-1} e^{(2\pi i/N)k_y n_y}\frac{\eta(k_x,k_y)}{(k_x^2+k_y^2)^{(H+1)/2}}. \tag{9}$$

From this expression, we readily construct the power spectrum along the cut $n_y=\text{const}$,

$$P_y(k_x)=|\tilde{h}(k_x,0)|^2+|\tilde{h}(N-k_x,0)|^2, \tag{10}$$

where for simplicity, and without loss of generality, we have set $n_y=0$. Using Eq. (7), we find

$$P_y(k_x)=\frac{2D}{N^2}\sum_{k=0}^{N-1}\left[\frac{1}{(k_x^2+k^2)^{1+H}}+\frac{1}{[k_x^2+(N-k)^2]^{1+H}}\right]. \tag{11}$$

For large N, this equation may be simplified to

$$P_y(k_x)=\frac{2D}{N^2}\frac{1}{k_x^{1+2H}}\int_0^{N/k_x}\frac{dz}{(1+z^2)^{1+H}}. \tag{12}$$

For $H>-1/2$, the integral in this equation approaches a constant rapidly as $N\to\infty$. However, for $H\leqslant-1/2$, it behaves as $(k_x/N)^{1+2H}$ for large N. Thus we conclude that

$$P_y(k_x)\sim\begin{cases}(1/k_x)^{1+2H} & \text{for } H>-1/2,\\ \text{const} & \text{for } H\leqslant-1/2.\end{cases} \tag{13}$$

This is precisely the behavior we see in Fig. 3. On the other hand, the power spectrum we find for the one-dimensional surface [Eq. (8)] is simply that of Eq. (2) irrespective of the Hurst exponent H.

One important lesson we draw from this problem and its resolution is that the Hurst exponent does *not* fully describe the correlations of self-affine surfaces: A two-dimensional surface with a given Hurst exponent may have completely different correlations from a one-dimensional surface, provided the Hurst exponent is low enough.

Another important, but related lesson, is that measuring the self-affine properties of a surface by averaging over one-dimensional cuts — which is the standard experimental approach — may lead to incorrect results. In fact, it was *knowing* the correct scaling of the stress field studied in Ref. [7], and comparing this with the measured quantities, that led to this work. Two-dimensional surfaces should preferably be analyzed using two-dimensional tools.

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[1] E. Bouchaud, J. Phys.: Condens. Matter **9**, 4319 (1997).
[2] A. L. Barabási and H. E. Stanley, *Fractal Growth Phenomena* (Cambridge University Press, Cambridge, 1997).
[3] J. Feder, *Fractals* (Plenum Press, New York, 1988).
[4] D. Tucotte, *Fractals and Chaos in Geology and Geophysics* (Cambridge University Press, New York, 1997).
[5] D. Vandembroucq and S. Roux, Phys. Rev. E **55**, 6186 (1997).
[6] I. Simonsen, D. Vandembroucq, and S. Roux, Phys. Rev. E **61**, 5914 (2000); e-print cond-mat/9907145.
[7] A. Hansen, J. Schmittbuhl, G. G. Batrouni, and F. A. Oliveira, e-print cond-mat/000519.
[8] J. Schmittbuhl, J. P. Vilotte, and S. Roux, Phys. Rev. E **51**, 131 (1995).
[9] M. Sahimi, Phys. Rep. **306**, 213 (1998).
[10] A. R. Mehrabi, H. Rassamdana, and M. Sahimi, Phys. Rev. E **56**, 712 (1997).
[11] I. Simonsen, A. Hansen, and O. M. Nes, Phys. Rev. E **58**, 2779 (1998).